RIEMANN-STIELTJES INTEGRAL OPERATORS BETWEEN WEIGHTED BERGMAN SPACES

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Abstract. This note completely describes the bounded or compact Riemann-Stieltjes integral operators T_g acting between the weighted Bergman space pairs (A^p_α, A^q_β) in terms of particular regularities of the holomorphic symbols g on the open unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : \ |z| < 1\}$ be the open unit ball of $\mathbb{C}^n$. Set $\partial\mathbb{B}_n = \{z \in \mathbb{C}^n : \ |z| = 1\}$ be the compact unit sphere of $\mathbb{C}^n$ – an n-dimensional Hilbert space over the complex field $\mathbb{C}$ under the inner product

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w_k}, \quad z = (z_1, z_2, ..., z_n),\ w = (w_1, w_2, ..., w_n) \in \mathbb{C}^n$$

and the associated norm

$$|z| = \langle z, z\rangle^{1/2}, \quad z \in \mathbb{C}^n.$$

Given a holomorphic map $g : \mathbb{B}_n \to \mathbb{C}$. For a holomorphic map $f : \mathbb{B}_n \to \mathbb{C}$, we, as studied in [Hu], define the Riemann-Stieltjes integral of f with respect to g via

$$T_g f(z) = \int_0^1 f(tz)\mathcal{R}g(tz)t^{-1}dt, \quad z \in \mathbb{B}_n \tag{1.1}$$

where

$$\mathcal{R}g(z) = \sum_{j=1}^{n} z_j \frac{\partial g(z)}{\partial z_j}, \quad z = (z_1, z_2, ..., z_n) \in \mathbb{B}_n$$

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stands for the radial derivative of g. In particular, if

$$\mathbf{1} = (1, 1, ..., 1) \quad \text{and} \quad g(z) = -\log(1 - \langle z, \mathbf{1}\rangle)$$

then (1.1) becomes

$$T_g f(z) = \int_0^1 f(tz)\langle tz, \mathbf{1}\rangle(1 - \langle tz, \mathbf{1}\rangle)^{-1} t^{-1} dt, \quad z \in \mathbb{B}_n,$$

a higher dimensional version of the classical Cesáro operator.

We consider the problem of determining the optimum on g such that

$$T_g : A^p_\alpha \to A^q_\beta \quad \textit{boundedly} \quad \textit{or} \quad \textit{compactly}. \tag{1.2}$$

Here and henceforth, for $p > 0$ and $\alpha > -1$, A^p_α is the weighted Bergman space of all holomorphic maps $f : \mathbb{B}_n \to \mathbb{C}$ satisfying

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p (1 - |z|^2)^\alpha dv(z)\right)^{1/p} < \infty,$$

where dv denotes the Lebesgue volume measure on $\mathbb{B}_n$. It is known that if $p = q \geq 1$ and $\alpha = \beta > -1$ then (1.2) holds if and only if g belongs to the Bloch space or the little Bloch space; see for example [AlSi] (for $n = 1$) and [Xi] (for $n \geq 1$). But, in other cases (even for $n = 1$), the optimal (A^p_α, A^q_β) estimates have not yet been worked out; see also [AlCi] (and references therein) for the setting of (H^p, H^q)–the 1-dimensional limit case of (A^p_α, A^q_β) as $\alpha = \beta \to -1$.

The goal of this note is to find out the optimal conditions of g such that (1.2) holds in all possible cases on $(p, q) \in (0, \infty) \times (0, \infty)$ and $(\alpha, \beta) \in (-1, \infty) \times (-1, \infty)$. Below is our result.

Theorem 1.1. *Let $\alpha, \beta \in (-1, \infty)$, $p, q \in (0, \infty)$ and $g : \mathbb{B}_n \to \mathbb{C}$ be holomorphic.*
(i) *If $p > q$ then $T_g : A^p_\alpha \to A^q_\beta$ boundedly or compactly if and only if*

$$\int_{\mathbb{B}_n} \left(|\mathcal{R}g(z)|(1 - |z|^2)^{1 - \frac{\alpha}{p} + \frac{\beta}{q}}\right)^{\frac{pq}{p-q}} dv(z) < \infty. \tag{1.3}$$

(ii) *If $p \leq q$ then $T_g : A^p_\alpha \to A^q_\beta$ boundedly or compactly if and only if*

$$\sup_{z \in \mathbb{B}_n} |\mathcal{R}g(z)|(1 - |z|^2)^{1 - \frac{n+1+\alpha}{p} + \frac{n+1+\beta}{q}} < \infty \tag{1.4}$$

or

$$\lim_{z \to \partial\mathbb{B}_n} |\mathcal{R}g(z)|(1 - |z|^2)^{1 - \frac{n+1+\alpha}{p} + \frac{n+1+\beta}{q}} = 0. \tag{1.5}$$

It is worth pointing out that the symbol g satisfying (1.3) or (1.4)/(1.5) is a constant whenever

$$1 \le \frac{1+\alpha}{p} - \frac{1+\beta}{q} \text{ and } p > q \quad \text{or} \quad \frac{n+1+\alpha}{p} - \frac{n+1+\beta}{q} > 1 \text{ and } p \le q.$$

The rest of this note is organized as follows. In Section 2, we collect some preliminary but useful facts on the weighted Bergman spaces and Khinchine's inequality. In Section 3, we demonstrate Theorem 1.1 through these preliminary results and some of the ideas exposed in [Lu] and [SmYa].

2. Preliminaries

When $p \ge 1$, the space A^p_α is a Banach space equipped with the norm $\|\cdot\|_{A^p_\alpha}$, and when $p \in (0,1)$, the space A^p_α is a complete metric space with the distance $d(f,g) = \|f-g\|^p_{A^p_\alpha}$.

First of all, we need a growth property of holomorphic functions on $\mathbb{B}_n$. To do so, we denote by ϕ_w the automorphism of $\mathbb{B}_n$ taking 0 to $w \in \mathbb{B}_n$, i.e.,

$$\phi_w(z) = \begin{cases} z, & w = 0 \\ \frac{w - P_w z - \sqrt{1-|z|^2} Q_w z}{1 - \langle z, w\rangle}, & w \neq 0, \end{cases}$$

where $Q_w = I - P_w$, I is the identity map and P_w is the projection of $\mathbb{C}^n$ onto the one-dimensional subspace spanned by $w \neq 0$. For $r > 0$ and $z \in \mathbb{B}_n$ put

$$D(z,r) = \left\{ w \in \mathbb{B}_n : \frac{1}{2} \log \frac{1 + |\phi_w(z)|}{1 - |\phi_w(z)|} < r \right\}$$

which is called the Bergman metric ball with center z and radius r.

Lemma 2.1. *Let $p \in (0,\infty)$ and $\alpha \in (-1,\infty)$. If $f : \mathbb{B}_n \to \mathbb{C}$ is holomorphic then*

$$|f(z)|(1-|z|^2)^{\frac{n+1+\alpha}{p}} \lesssim \left(\int_{D(z,r)} |f(w)|^p (1-|w|^2)^\alpha dv(w) \right)^{\frac{1}{p}}, \quad z \in \mathbb{B}_n. \tag{2.1}$$

For a proof of (2.1), see also [Zh, p. 64].

Next, we state two characterizations of the weighted Bergman spaces.

Lemma 2.2. *Let $p \in (0,\infty)$, $\alpha \in (-1,\infty)$, and $f : \mathbb{B}_n \to \mathbb{C}$ be holomorphic. Then the following three statements are equivalent:*

(i) $f \in A^p_\alpha$.

(ii)

$$\|\!|f|\!\|_{A^p_\alpha} = |f(0)| + \left(\int_{\mathbb{B}_n} |\mathcal{R}f(z)|^p (1-|z|^2)^{p+\alpha} dv(z)\right)^{\frac{1}{p}} < \infty.$$

(iii) *For any* $\eta \in (0,1]$, *there exist a sequence* $\{z_j\}$ *in* $\mathbb{B}_n$ *such that*

(a) $\mathbb{B}_n = \cup_j D(z_j, \eta)$;

(b) $D(z_j, \frac{\eta}{4}) \cap D(z_k, \frac{\eta}{4}) = \emptyset$ *for* $j \neq k$;

(c) *Each point* $z \in \mathbb{B}_n$ *lies in at most* $N = N(\eta)$ *of balls from* $\{D(z_j, 2\eta)\}$;

(d)

$$f(z) = \sum_j c_j \frac{(1-|z_j|^2)^{\frac{pb-n-1-\alpha}{p}}}{(1-\langle z, z_j\rangle)^b}, \quad z \in \mathbb{B}_n,$$

where $\{c_j\}$ *is in the sequence space* l^p *and* b *is a constant greater than* $n \max\{1, \frac{1}{p}\} + \frac{1+\alpha}{p}$. *Moreover, if* $f \in A^p_\alpha$ *then*

$$\|\!|f|\!\|_{A^p_\alpha} \approx \|f\|_{A^p_\alpha} \approx \|\{c_j\}\|_{l^p}. \tag{2.2}$$

For a proof of Lemma 2.2 and its sources, see for example [Zh, Chapter 2] and [CoRo] as well as [Sh].

Finally, we quote the following well-known form of Khinchine's inequality.

Lemma 2.3. *Suppose*

$$r_0(t) = \begin{cases} 1, & 0 \le t - [t] < 1/2 \\ -1, & 1/2 \le t - [t] < 1 \end{cases}$$

and

$$r_j(t) = r_0(2^j t), \quad j = 1, 2, ...,$$

and let $p \in (0, \infty)$ *and* $(c_1, ..., c_m) \in \mathbb{C}^m$, $m = 1, 2, ...$ *Then*

$$\left(\sum_{j=1}^m |c_j|^2\right)^{\frac{1}{2}} \approx \left(\int_0^1 \Big|\sum_{j=1}^m c_j r_j(t)\Big|^p dt\right)^{\frac{1}{p}}. \tag{2.3}$$

Note: In the above (and below), the notation $U \approx V$ means that there are two constants $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 V \le U \le \kappa_2 V$. Moreover, if $U \le \kappa_2 V$ then we say $U \lesssim V$.

3. Proof

To begin with, we notice that the formula

$$\mathcal{R}T_g f(z) = f(z)\mathcal{R}g(z), \quad z \in \mathbb{B}_n, \tag{3.1}$$

holds for all holomorphic maps $f, g : \mathbb{B}_n \to \mathbb{C}$. And, let us agree to two more conventions:

$$dv_\beta(z) = (1-|z|^2)^\beta dv(z), \quad z \in \mathbb{B}_n,$$

and

$$\|T_g\|_{A^p_\alpha \to A^q_\beta} = \sup_{\|f\|_{A^p_\alpha}=1} \|T_g f\|_{A^q_\beta}.$$

It is clear that

$$\|T_g f\|_{A^q_\beta} \le \|T_g\|_{A^p_\alpha \to A^q_\beta} \|f\|_{A^p_\alpha}, \quad f \in A^p_\alpha. \tag{3.2}$$

Proof of Theorem 1.1 (i). Suppose $p > q$. Let (1.3) be true. By (2.2), (3.1) and Hölder's inequality we have that if $f \in A^p_\alpha$ then

$$\begin{aligned}\|T_g f\|^q_{A^q_\beta} &\approx \int_{\mathbb{B}_n} |f(z)\mathcal{R}g(z)|^q (1-|z|^2)^q dv_\beta(z) \\ &\lesssim \|f\|^q_{A^p_\alpha} \left(\int_{\mathbb{B}_n} \left(|\mathcal{R}g(z)|(1-|z|^2)^{1-\frac{\alpha}{p}+\frac{\beta}{q}} \right)^{\frac{pq}{p-q}} dv(z) \right)^{\frac{p-q}{p}},\end{aligned}$$

implying the boundedness of $T_g : A^p_\alpha \to A^q_\beta$.

Conversely, suppose $T_g : A^p_\alpha \to A^q_\beta$ is bounded. Then $\|T_g\|_{A^p_\alpha \to A^q_\beta}$ is finite with (3.2). For each natural number j let

$$K_j(z) = \frac{(1-|z_j|^2)^{\frac{pb-n-1-\alpha}{p}}}{(1-\langle z, z_j \rangle)^b},$$

where $\{z_j\}$ is the sequence in Lemma 2.2 (iii). Let $\{c_j\} \in l^p$, and choose $\{r_j(t)\}$ as obeying Lemma 2.3. Then $\{c_j r_j(t)\} \in l^p$ with $\|\{c_j r_j(t)\}\|_{l^p} = \|\{c_j\}\|_{l^p}$, and so $\sum_j c_j r_j(t) K_j \in A^p_\alpha$ with

$$\Big\| \sum_j c_j r_j(t) K_j \Big\|_{A^p_\alpha} \approx \|\{c_j\}\|_{l^p},$$

due to Lemma 2.2 (iii). This fact plus (3.2) derives

$$\int_{\mathbb{B}_n} \Big| T_g \Big(\sum_j c_j r_j(t) K_j \Big)(z) \Big|^q dv_\beta(z) \lesssim \|T_g\|^q_{A^p_\alpha \to A^q_\beta} \|\{c_j\}\|^q_{l^p},$$

Furthermore, integrating this inequality from 0 to 1 with respect to dt, as well as using (3.1), Fubini's theorem and (2.3) in Lemma 2.3, we get

$$\int_{\mathbb{B}_n}\Big(\sum_j \big|c_jK_j(z)\big|^2\Big)^{\frac{q}{2}}\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)\lesssim \|T_g\|^q_{A^p_\alpha\to A^q_\beta}\|\{c_j\}\|^q_{l^p}.$$

Noticing the estimate

$$|1-\langle z,z_j\rangle|\approx 1-|z_j|^2\quad\text{as}\quad z\in D(z_j,2\eta),$$

applying the condition (c) in Lemma 2.2 (iii), and letting 1_E be the characteristic function of a set $E\subseteq\mathbb{B}_n$, we achieve

$$\begin{aligned}
&\sum_j|c_j|^q\left(\frac{\int_{D(z_j,2\eta)}\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)}{(1-|z_j|^2)^{\frac{q(n+1+\alpha)}{p}}}\right)\\
&\quad=\int_{\mathbb{B}_n}\sum_j\left(\frac{|c_j|^q 1_{D(z_j,2\eta)}(z)}{(1-|z_j|^2)^{\frac{q(n+1+\alpha)}{p}}}\right)\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)\\
&\quad\lesssim\int_{\mathbb{B}_n}\left(\sum_j\frac{|c_j|^2 1_{D(z_j,2\eta)}(z)}{(1-|z_j|^2)^{\frac{2(n+1+\alpha)}{p}}}\right)^{\frac{q}{2}}\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)\\
&\quad\lesssim\int_{\mathbb{B}_n}\Big(\sum_j|c_jK_j(z)|^2\Big)^{\frac{q}{2}}\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)\\
&\quad\lesssim\|T_g\|^q_{A^p_\alpha\to A^q_\beta}\|\{c_j\}\|^q_{l^p}.
\end{aligned}$$

The last estimate indicates

$$\left\{\frac{\int_{D(z_j,2\eta)}\big(|\mathcal{R}g(z)|(1-|z|^2)\big)^q dv_\beta(z)}{(1-|z_j|^2)^{\frac{q(n+1+\alpha)}{p}}}\right\}\in l^{\frac{p}{p-q}}.$$

Because $\mathcal{R}g$ is holomorphic on $\mathbb{B}_n$, by (2.2), (2.1) and the condition (a) in Lemma 2.2 (iii)

we achieve

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^{\frac{pq}{p-q}} (1-|z|^2)^{\frac{\beta p-\alpha q}{p-q}} dv(z) \\
&\quad \lesssim \sum_j \int_{D(z_j,\eta)} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^{\frac{pq}{p-q}} (1-|z|^2)^{\frac{\beta p-\alpha q}{p-q}} dv(z) \\
&\quad \lesssim \sum_j \int_{D(z_j,\eta)} \left(\frac{\int_{D(z,\eta)} \left(|\mathcal{R}g(w)|(1-|w|^2)\right)^q dv_\beta(w)}{(1-|z|^2)^{\frac{q\alpha+p(n+1)}{p}}} \right)^{\frac{p}{p-q}} dv(z) \\
&\quad \lesssim \sum_j \left(\frac{\int_{D(z_j,2\eta)} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^q dv_\beta(z)}{(1-|z_j|^2)^{\frac{q\alpha+p(n+1)}{p}}} \right)^{\frac{p}{p-q}} (1-|z_j|^2)^{n+1} \\
&\quad \lesssim \sum_j \left(\frac{\int_{D(z_j,2\eta)} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^q dv_\beta(z)}{(1-|z_j|^2)^{\frac{q(n+1+\alpha)}{p}}} \right)^{\frac{p}{p-q}} \\
&\quad \lesssim \|T_g\|^{\frac{pq}{p-q}}_{A^p_\alpha \to A^q_\beta},
\end{aligned}
$$

giving (1.3).

Regarding the compactness, it suffices to show that if (1.3) holds then $T_g : A^p_\alpha \to A^q_\beta$ is compact. Assuming (1.3), we obtain that T_g is a bounded operator from $A^p_\alpha \to A^q_\beta$ and so $g = g(0) + T_g 1 \in A^q_\beta$. In addition to this, we also see that for any $\epsilon > 0$ there is a $\delta \in (0,1)$ such that

$$
\int_{|z|>\delta} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^{\frac{pq}{p-q}} (1-|z|^2)^{\frac{\beta p-\alpha q}{p-q}} dv(z) < \epsilon.
$$

Since the weak convergence in A^p_α means the uniform convergence on compacta of $\mathbb{B}_n$, we may assume that $\{f_j\}$ is any sequence in the unit ball of A^p_α and converges to 0 uniformly on compacta of $\mathbb{B}_n$. For the above $\epsilon > 0$ there exists an integer $j_0 > 0$ such that $\sup_{|z|\le\delta} |f_j(z)| < \epsilon$ as $j \ge j_0$. With the help of (3.1), (2.2) and Hölder's inequality, we further obtain

$$
\begin{aligned}
\|T_g f_j\|^q_{A^q_\beta} &\approx \left(\int_{|z|\le\delta} + \int_{|z|>\delta} \right) \left(|f_j(z)\mathcal{R}g(z)|(1-|z|^2)\right)^q dv_\beta(z) \\
&\lesssim \epsilon^q \|g\|^q_{A^q_\beta} + \|f_j\|^q_{A^p_\alpha} \left(\int_{|z|>\delta} \left(|\mathcal{R}g(z)|(1-|z|^2)\right)^{\frac{pq}{p-q}} (1-|z|^2)^{\frac{\beta p-\alpha q}{p-q}} dv(z) \right)^{\frac{p-q}{p}} \\
&\lesssim \epsilon^q \|g\|^q_{A^q_\beta} + \epsilon^{\frac{p-q}{p}}.
\end{aligned}
$$

In other words, $\lim_{j\to\infty} \|T_g f_j\|_{A^q_\beta} = 0$ and so $T_g : A^p_\alpha \to A^q_\beta$ is compact.

Proof of Theorem 1.1 (ii). Suppose now $p \le q$. If (1.4) holds then

$$\|g\|_{B^\gamma} = \sup_{z\in\mathbb{B}_n} |\mathcal{R}g(z)|(1-|z|^2)^\gamma < \infty, \quad \gamma = 1 - \frac{n+1+\alpha}{p} + \frac{n+1+\beta}{q}.$$

From (3.1), (2.2) and (2.1) it turns out that for $f \in A^p_\alpha$,

$$\begin{aligned}\|T_g f\|^q_{A^q_\beta} &\approx \int_{\mathbb{B}_n} |f(z)|^p |f(z)|^{q-p} |\mathcal{R}g(z)|^q (1-|z|^2)^q dv_\beta(z)\\ &\lesssim \|f\|^{q-p}_{A^p_\alpha} \|g\|^q_{B^\gamma} \int_{\mathbb{B}_n} |f(z)|^p (1-|z|^2)^{q-q\gamma-\frac{(q-p)(n+1+\alpha)}{p}} dv_\beta(z)\\ &\lesssim \|f\|^q_{A^p_\alpha} \|g\|^q_{B^\gamma}.\end{aligned}$$

That is to say, $T_g : A^p_\alpha \to A^q_\beta$ is bounded.

Conversely, if $T_g : A^p_\alpha \to A^q_\beta$ is bounded then the operator norm $\|T_g\|_{A^p_\alpha \to A^q_\beta}$ is finite with (3.2). Keeping this in mind, we deal with two cases: $p > 1$ and $p \le 1$.

Case 1: $p > 1$. Define

$$K(w,z) = (1 - \langle z,w\rangle)^{-(n+\alpha+1)}, \quad z,w \in \mathbb{B}_n.$$

A routine calculation (see for example [FaKo] or [Zh, pp. 20-21]) yields

$$\|K(w,\cdot)\|_{A^p_\alpha} \approx (1-|w|^2)^{-\frac{(n+1+\alpha)(p-1)}{p}}. \tag{3.3}$$

Using (3.1), (2.2), certain transformation properties of ϕ_w and (2.1) (for $\mathcal{R}g(\phi_w)$), we obtain

$$\begin{aligned}\|T_g K(w,\cdot)\|^q_{A^q_\beta} &\approx \int_{\mathbb{B}_n} |K(w,z)|^q |\mathcal{R}g(z)|^q (1-|z|^2)^q dv_\beta(z)\\ &\gtrsim (1-|w|^2)^{q+(1-q)(n+1+\alpha)+\beta-\alpha} \int_{|u|\le 1/2} \frac{|\mathcal{R}g(\phi_w(u))|^q (1-|u|^2)^{q+\beta}}{|1-\langle u,w\rangle|^{2n+2-(n+1+\alpha)q+2(\beta-\alpha)}} dv(u)\\ &\gtrsim (1-|w|^2)^{q+(1-q)(n+1+\alpha)+\beta-\alpha} |\mathcal{R}g(\phi_w(0))|^q\\ &\gtrsim (1-|w|^2)^{q+(1-q)(n+1+\alpha)+\beta-\alpha} |\mathcal{R}g(w)|^q.\end{aligned}$$

In brief, we have

$$\|T_g K(w,\cdot)\|^q_{A^q_\beta} \gtrsim (1-|w|^2)^{q+(1-q)(n+1+\alpha)+\beta-\alpha} |\mathcal{R}g(w)|^q. \tag{3.4}$$

This estimate, together with (3.3) and (3.2) (for $f(\cdot) = K(w,\cdot)$), produces (1.4) right away.

Case 2: $p \le 1$. Select a positive integer $m > n+1+\alpha$ and set

$$K_p(w,z) = \big(1 - \langle z, w\rangle\big)^{-\frac{m}{p}}, \quad z, w \in \mathbb{B}_n.$$

Just like the case of $p > 1$, it follows that

$$\|K_p(w,\cdot)\|_{A^p_\alpha} \approx (1-|w|^2)^{\frac{n+1+\alpha-m}{p}} \tag{3.5}$$

and

$$\|T_g K_p(w,\cdot)\|^q_{A^q_\beta} \gtrsim (1-|w|^2)^{q+n+1+\beta-\frac{mq}{p}} |\mathcal{R}g(w)|^q. \tag{3.6}$$

A combination of (3.6), (3.5) and (3.2) (for $f(\cdot) = K_p(w,\cdot)$) yields (1.4) too.

To establish the corresponding compactness part, we assume that g satisfies (1.5). Then $g \in A^q_\beta$, and for any $\epsilon > 0$ there is an $\delta \in (0,1)$ such that as $|z| \in (\delta, 1)$,

$$|\mathcal{R}g(z)|(1-|z|^2)^\gamma < \epsilon, \quad \gamma = 1 - \frac{n+1+\alpha}{p} + \frac{n+1+\beta}{q}.$$

In order to prove that $T_g : A^p_\alpha \to A^q_\beta$ is compact, we consider any sequence $\{f_j\}$ in the unit ball of A^p_α which converges to 0 uniformly on compacta of $\mathbb{B}_n$. For such a sequence, there is an integer $j_0 > 0$ such that $\sup_{|z|\le\delta} |f_j(z)| < \epsilon$ when $j \ge j_0$. Hence by (3.1), (2.1) and (2.2),

$$\begin{aligned}
\|T_g f_j\|^q_{A^q_\beta} &\approx \int_{\mathbb{B}_n} |f_j(z)\mathcal{R}g(z)|^q (1-|z|^2)^q dv_\beta(z) \\
&\lesssim \epsilon^q \int_{|z|\le\delta} |\mathcal{R}g(z)|^q(1-|z|^2)^q dv_\beta(z) + \epsilon^q \|f_j\|^{q-p}_{A^p_\alpha} \int_{|z|>\delta} |f_j(z)|^p (1-|z|^2)^\alpha dv(z) \\
&\lesssim \epsilon^q \Big(\|g\|^q_{A^q_\beta} + \|f_k\|^q_{A^p_\alpha} \Big) \\
&\lesssim \epsilon^q \Big(\|g\|^q_{A^q_\beta} + 1 \Big).
\end{aligned}$$

Namely, $\|T_g f_j\|_{A^q_\beta} \to 0$ as $j \to \infty$. Therefore T_g is a compact operator from A^p_α to A^q_β.

On the other hand, if $T_g : A^p_\alpha \to A^q_\beta$ is compact, then, for $z, w \in \mathbb{B}_n$ let

$$k_p(w,z) = \begin{cases} \frac{K(w,z)}{\|K(w,\cdot)\|_{A^p_\alpha}}, & p > 1 \\ \frac{K_p(w,z)}{\|K_p(w,\cdot)\|_{A^p_\alpha}}, & p \le 1. \end{cases}$$

Obviously, $k_p(w,\cdot)$ tend to 0 uniformly on compacta of $\mathbb{B}_n$ as $w \to \partial\mathbb{B}_n$. By the compactness of $T_g : A^p_\alpha \to A^q_\beta$, we find

$$\lim_{w\to\partial\mathbb{B}_n} \|T_g k_p(w,\cdot)\|_{A^q_\beta} = 0.$$

The above limit, along with (3.3), (3.4), (3.5) and (3.6), yield (1.5).

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